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Integrating Functions of Two Variables

Anti-Partial-Derivatives

$$\int f(x,y)dx = F(x,y) + C \qquad \text{if and only if} \qquad \frac{\partial}{\partial x}F(x,y) = f(x,y)$$

$$\int f(x,y)dy = F(x,y) + C \qquad \text{if and only if} \qquad \frac{\partial}{\partial y}F(x,y) = f(x,y)$$

Examples:

1. Compute $\int xy^3 dx$. Check your answer by computing $\frac{\partial}{\partial x}\left[\int xy^3 dx\right]$.

$\int xy^3 dx = \frac{x^2}{2}y^3 + c.$

check: $\frac{\partial}{\partial x}\left[\frac{x^2y^3}{2} + c\right] = \frac{\cancel{2}}{\cancel{2}}xy^3 + 0$ ✓

2. Compute $\int xy^3 dy$. Check your answer by computing $\frac{\partial}{\partial y}\left[\int xy^3 dx\right]$.

$\int xy^3 dy = x\frac{y^4}{4} + c$

check: $\frac{\partial}{\partial y}\left[x\frac{y^4}{4} + c\right] = \frac{x\cdot\cancel{4}y^3}{\cancel{4}} + 0$

Iterated Integrals

Compute Iterated integrals from the inside out.

Examples:

1. Compute $\int\left[\int xy^3\, dx\right]dy. = \int \frac{x^2}{2}y^3 dy = \frac{x^2y^4}{2\cdot 4} + C$

2. Compute $\int\left[\int xy^3\, dy\right]dx. = \int x\frac{y^4}{4} dx = \frac{x^2}{2}\cdot\frac{y^4}{4} + C$

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Double Integrals over Rectangles

Notation: we denote rectangles as the *cartesian* product of two intervals.

$$R = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b \text{ and } c \leq y \leq d\}$$

1. Let $R = [0, 3] \times [0, 2]$. Compute the following integral

$$\iint_R xy^3 \, dA$$

$$= \int_0^3 \int_0^2 xy^3 \, dy \, dx = \int_0^3 \left[x \frac{y^4}{4} \right]_0^2 dx$$

$$= \int_0^3 \left[x \frac{2^4}{2^2} - x \cdot \frac{0^4}{2^2} \right] dx$$

$$= \int_0^3 4x \, dx = \left.\frac{\cancel{4}x^2}{\cancel{2}}\right]_0^3 = 2 \cdot 3^2 - 2 \cdot 0^2$$

$$= 18$$

2. Let $R = [1, 2] \times [0, \frac{\pi}{3}]$. Compute the following integral

$$\iint_R x \cos(2y) \, dA$$

$$= \int_1^2 \left[\int_0^{\pi/3} x \cdot \cos(2\underline{\underline{y}}) \, d\underline{\underline{y}} \right] dx = \int_1^2 \left[x \cdot \frac{\sin(2y)}{2} \right]_0^{\pi/3} dx$$

$$= \int_1^2 \left[\frac{x \cdot \sin(\frac{2\pi}{3})}{2} - \underbrace{\frac{x \cdot \sin(0)}{2}}_{=0} \right] dx$$

$$\sin\left(\tfrac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$= \left[\frac{x^2}{2} \cdot \frac{\sin(\frac{2\pi}{3})}{2} \right]_1^2 = \frac{2^2}{2} \cdot \frac{\frac{\sqrt{3}}{2}}{2} - \frac{1^2}{2} \cdot \frac{\frac{\sqrt{3}}{2}}{2}$$

$$\boxed{= \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}}$$

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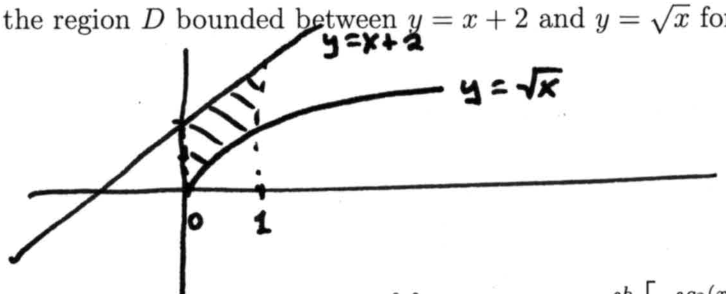
Double Integrals over More General Regions

There are two main types of non-rectangular regions D.

Type 1: The top and bottom boundaries (the y boundaries) are given as functions of x.

$$\text{Formally}: \; D = \Big\{(x,y) : a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\Big\}$$

E.g. Sketch the region D bounded between $y = x + 2$ and $y = \sqrt{x}$ for x between 0 and 1.



$$\text{Net volume under } f \text{ on } D = \iint_D f(x,y)dA = \int_a^b \underbrace{\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\right]} dx$$

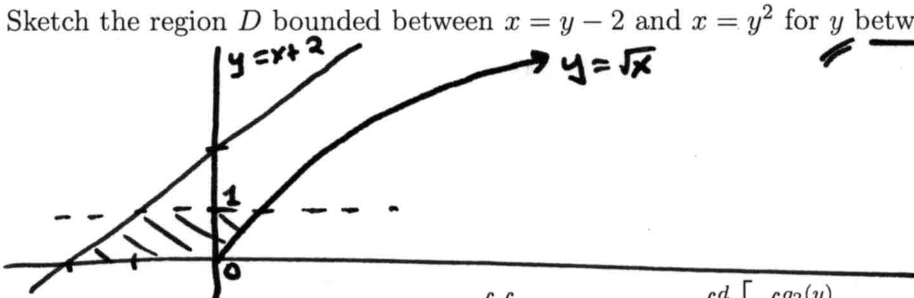
Idea: the innermost integral $\displaystyle\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy$

gives the "area under f above the slice at x between $y = g_1(x)$ to $y = g_2(x)$."

Type 2: The left and right boundaries (the x boundaries) are given as functions of y.

$$\text{Formally}: \; D = \Big\{(x,y) : g_1(y) \le x \le g_2(y) \text{ and } c \le y \le d\Big\}$$

E.g. Sketch the region D bounded between $x = y - 2$ and $x = y^2$ for y between 0 and 1.



$$\text{Net volume under } f \text{ on } D = \iint_D f(x,y)dA = \int_c^d \underbrace{\left[\int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\right]} dy$$

Idea: the innermost integral $\displaystyle\int_{g_1(y)}^{g_2(y)} f(x,y)\, dx$

gives the "area under f above the slice at y between $x = g_1(y)$ to $x = g_2(y)$."

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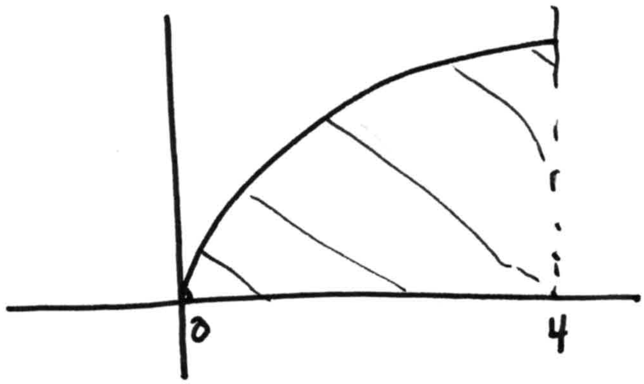
1. Let $\displaystyle\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{x}} xy\,dy\,dx$

   (a) Sketch the region of integration and (b) compute the specified volume.



$$\int_0^4\left(\int_0^{\sqrt{x}} xy\,dy\right)dx = \int_0^4 \left[x\,y^2\right]_0^{\sqrt{x}}\,dx$$

$$= \int_0^4 x\cdot\underbrace{(\sqrt{x})^2}_{x} - x\cdot 0^2\,dx$$

$$= \int_0^4 x^2\,dx$$

$$= \left.\frac{x^3}{3}\right]_0^4$$

$$= \frac{4^3}{3} - \frac{0^3}{3}$$

$$= \frac{64}{3}$$

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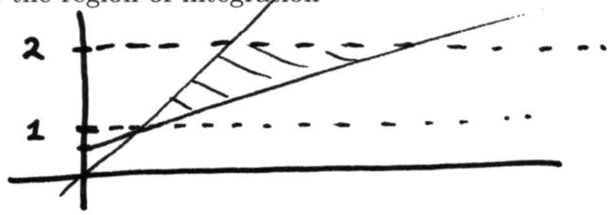
$x = 2y - 1 \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$

$x = y \Rightarrow y = x$

2. Let $\int_1^2 \left[\int_y^{2y-1} 2x - y^2 \, dx \right] dy$

(a) Sketch the region of integration



(b) Compute the specified volume.

$$\int_1^2 \left[\int_y^{2y-1} 2x - y^2 \, dx \right] dy$$

$$= \int_1^2 \left[\frac{\cancel{2}x^2}{\cancel{2}} - y^2 x \right]_y^{2y-1} dy$$

$$= \int_1^2 \left((2y-1)^2 - y^2(2y-1) \right) - \left(y^2 - y^2 \cdot y \right) dy$$

$$= \int_1^2 4y^2 - 4y + 1 - 2y^3 + \cancel{y^2} - \cancel{y^2} + y^3 \, dy$$

$$= \int_1^2 4y^2 - 4y + 1 - y^3 \, dy$$

$$= \frac{4y^3}{3} - \frac{4y^2}{2} + y - \frac{y^4}{4} \Big]_1^2$$

$$= \left(\frac{4 \cdot 2^3}{3} - \frac{4 \cdot 2^2}{2} + 2 + \frac{2^4}{4} \right) - \left(\frac{4 \cdot 1}{3} - \frac{4 \cdot 1}{2} + 1 - \frac{1}{4} \right)$$

$= \ldots$

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3. Compute the volume under $f(x,y) = y - x^2$ on the region bounded between $y = 3x$ and $y = x^2$ on the interval where x is in $[0,2]$

$$\iint y-x^2\,dA = \int_0^2\left[\int_{x^2}^{3x} y-x^2\,dy\right]dx$$

$$= \int_0^2 \left[\frac{y^2}{2} - x^2y\right]_{x^2}^{3x} dx$$

$$= \int_0^2 \left(\frac{9x^2}{2} - x^2\cdot 3x\right) - \left(\frac{x^4}{2} - x^2\cdot x^2\right) dx$$

$$= \int_0^2 \frac{9}{2}x^2 - 3x^3 + \frac{1}{2}x^4\,dx$$

$$= \left[\frac{\overset{3}{\cancel{9}}}{2}\cancel{\frac{x^3}{3}} - 3\frac{x^4}{4} + \frac{1}{2}\frac{x^5}{5}\right]_0^2$$

$$= \left(\frac{3\cdot 2^2}{2} - 3\cdot\frac{2^4}{4} + \frac{1}{2}\cdot\frac{2^5}{5}\right) - (0 - 0 + 0)$$

$$= 3\cdot 2 - 3\cdot 4 + \frac{16}{5}$$

$$= -6 + \frac{16}{5}$$

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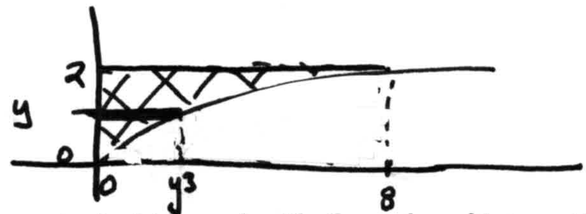
4. Let $\int_{x=0}^{x=8}\int_{y=x^{1/3}}^{y=2} \frac{1}{y^4+1}\,dy\,dx$.

(a) Sketch the region of integration



(b) Write an *equivalent* integral with the order of integration reversed

area of slice at y $= \int_0^{y^3} \frac{1}{y^4+1}\,dx$

net volume = integrate y slices $= \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$

(c) Compute the volume under $f(x,y) = \frac{1}{y^4+1}$ on the specified region

$$= \int_0^2 \left[\frac{1}{y^4+1}\,x\right]_0^{y^3} dy$$

$$= \int_0^2 \frac{1}{\underbrace{y^4+1}_{u}} \cdot \underbrace{\boxed{y^3\,dy}}_{\frac{du}{4}}$$

$u = y^4+1$

$\frac{du}{dy} = 4y^3$

$\frac{du}{4} = y^3\,dy$

$y=0 \Rightarrow u = 0+1 = 1$

$y=2 \Rightarrow u = 16+1 = 17$

$$= \int_1^{17} \frac{1}{u}\cdot\frac{du}{4} = \frac{1}{4}\cdot \ln|u|\Big]_1^{17} = \frac{\ln(17)}{4} - \frac{\ln(1)}{4}$$

$$= \boxed{\frac{\ln(17)}{4}}$$